Correlation function of the two-dimensional Ising model on a finite lattice. II

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Abstract

We calculate the two-point correlation function and magnetic susceptibility in the anisotropic 2D Ising model on a lattice with one infinite and the other finite dimension, along which periodic boundary conditions are imposed. Using exact expressions for a part of lattice form factors, we propose the formulas for arbitrary spin matrix elements, thus providing a possibility to compute all multipoint correlation functions in the anisotropic Ising model on cylindrical and toroidal lattices. The scaling limit of the corresponding expressions is also analyzed.

1 The model

Two-point correlation function and magnetic susceptibility of the isotropic Ising model on a cylinder were calculated in [1, 2]. Analogous results can be obtained in the anisotropic case as well. The complications due to the presence of two different coupling constants can be overcome, since the matrix whose determinant yields the correlation function continues to have a Toeplitz form. The computation idea has already been presented in [1], and below we will often use the results of that paper, omitting detailed calculations.

The calculation of multipoint correlation functions can be reduced to the problem of finding Ising spin matrix elements (the so-called form factors) in the orthonormal basis of transfer matrix eigenstates. Although neither the first nor the second problem has been solved on a finite lattice, the language of matrix elements turns out to be more convenient when constructing the corresponding expressions. A part of form factors can be found from the formulas for the two-point correlation function. Since the structure of these expressions is relatively simple, we generalize them to the case of arbitrary matrix elements.

The hamiltonian of the anisotropic Ising model on a rectangular lattice is defined as

$$H[\sigma] = -\sum_{\mathbf{r}} \sigma(\mathbf{r}) \left(J_x \nabla_x + J_y \nabla_y \right) \sigma(\mathbf{r}),$$

where the two-dimensional vector $\mathbf{r} = (r_x, r_y)$ labels the lattice sites: $r_x = 1,\ 2,\ \ldots,\ M$, $r_y = 1,\ 2,\ \ldots,\ N$; the spins $\sigma(\mathbf{r})$ take on the values ± 1. The parameters J_x and J_y

determine the coupling energies of adjacent spins in the horizontal and vertical direction. The operators of shifts by one lattice site, ∇_x and ∇_y, are given by

$$\nabla_x \sigma(r_x, r_y) = \sigma(r_x + 1, r_y), \quad \nabla_y \sigma(r_x, r_y) = \sigma(r_x, r_y + 1),$$

where for periodic boundary conditions one has

$$\left(\nabla_x^{(R)}\right)^M = \left(\nabla_y^{(R)}\right)^N = 1,$$

and for antiperiodic ones

$$\left(\nabla_x^{(NS)}\right)^M = \left(\nabla_y^{(NS)}\right)^N = -1.$$

If the lattice is periodic in both directions, the partition function of the model at the temperature β^{-1}

$$Z = \sum_{[\sigma]} e^{-\beta H[\sigma]}$$

can be written as a sum of four terms

$$Z = \frac{1}{2}\left(Q^{(NS,\,NS)} + Q^{(NS,\,R)} + Q^{(R,\,NS)} - Q^{(R,\,R)}\right), \tag{1}$$

each of them being proportional to the pfaffian of the operator

$$\hat{D} = \begin{pmatrix} 0 & 1 & 1 + t_x \nabla_x & 1 \\ -1 & 0 & 1 & 1 + t_y \nabla_y \\ -1 - t_x \nabla_{-x} & -1 & 0 & 1 \\ -1 & -1 - t_y \nabla_{-y} & -1 & 0 \end{pmatrix},$$

$$Q = (2\cosh K_x \cosh K_y)^{MN} \cdot \mathrm{Pf}\, \hat{D},$$

with different boundary conditions for ∇_x, ∇_y. Here, we have introduced the dimensionless parameters

$$K_x = \beta J_x, \quad K_y = \beta J_y, \quad t_x = \tanh K_x, \quad t_y = \tanh K_y.$$

One can verify that when the torus degenerates into a cylinder ($M \gg N$), the partition function is determined by the antiperiodic term: $Z = Q^{(NS,\,NS)}$.

Let us consider the two-point correlation function in the Ising model on the cylinder

$$\langle \sigma(0,0)\sigma(r_x, r_y)\rangle = Z^{-1} \sum_{[\sigma]} \sigma(0,0)\sigma(r_x, r_y) \mathrm{e}^{-\beta H[\sigma]} = i^P \frac{Z_{def}}{Z} = i^P \frac{Q_{def}^{(NS,\,NS)}}{Q^{(NS,\,NS)}}, \tag{2}$$

where Z_{def} denotes the partition function of the Ising model with a defect: the coupling parameters K_x, K_y should be replaced by $K_x - i\pi/2$, $K_y - i\pi/2$ along a path that connects the correlating spins (see Fig. 1 in [1], where the numbering of lattice sites and the locations of correlating spins were described; the bold line in the figure is the path along which the couplings are modified). The exponent P in the formula (2) is equal to the number of steps

along the defect line: $P = r_x + r_y$ in the case of a shortest path. When the correlating spins are located along a line parallel to the cylinder axis (i. e. $r_y = 0$), the ratio of pfaffians in the right hand side of (2) can be expressed in terms of the determinant of a Toeplitz matrix

$$\langle\sigma(0,0)\sigma(r_x,0)\rangle = \det A, \tag{3}$$

$$A_{kk'} = \int\limits_{-\pi}^{\pi} \frac{dp\; e^{ip(k'-k)}}{2\pi N} \sum_q{}^{(NS)} \frac{2t_x\left(1+t_y^2\right) + \left(t_y^2-1\right)\left(e^{-ip}+t_x^2 e^{ip}\right)}{\left(1+t_x^2\right)\left(1+t_y^2\right) - 2t_x\left(1-t_y^2\right)\cos p - 2t_y\left(1-t_x^2\right)\cos q}, \tag{4}$$

whose size $|r_x| \times |r_x|$ is determined by the distance between the correlating spins. Here and below the superscripts (NS) and (R) in sums and products imply that the corresponding operations are performed with respect to Neveu-Schwartz ($q = \frac{2\pi}{N}\left(j+\frac{1}{2}\right)$, $j = 0, 1, \ldots, N-1$) or Ramond ($q = \frac{2\pi}{N} j$, $j = 0, 1, \ldots, N-1$) values of quasimomenta. Our goal is to transform (3)–(4) into a representation with an explicit dependence on the distance.

2 Ferromagnetic phase

In the translationally invariant case the pfaffians $\mathrm{Pf}\,\hat{D}$ can be easily calculated,

$$Q = \prod_{q,p} (\cosh 2K_x \cosh 2K_y - \sinh 2K_x \cos q - \sinh 2K_y \cos p)^{1/2}. \tag{5}$$

The product over any of the two quasimomentum components in (5) can be found in an explicit form. For instance, the term in (1), which corresponds to periodic boundary conditions along the x axis and antiperiodic ones along the y axis, may be written as

$$\begin{aligned} Q^{(R,NS)} &= (2\sinh 2K_y)^{MN/2} \prod_q{}^{(R)} \mathrm{e}^{N\gamma(q)/2}(1+\mathrm{e}^{-N\gamma(q)}) = \\ &= (2\sinh 2K_x)^{MN/2} \prod_q{}^{(NS)} \mathrm{e}^{M\overline{\gamma}(p)/2}(1-\mathrm{e}^{-M\overline{\gamma}(p)}), \end{aligned} \tag{6}$$

where the functions $\gamma(q)$ and $\overline{\gamma}(p)$ are determined by the relations

$$\cosh\gamma(q) = \frac{(1+t_x^2)(1+t_y^2)}{2t_x(1-t_y^2)} - \frac{t_y(1-t_x^2)}{t_x(1-t_y^2)}\cos q, \tag{7}$$

$$\cosh\overline{\gamma}(p) = \frac{(1+t_x^2)(1+t_y^2)}{2t_y(1-t_x^2)} - \frac{t_x(1-t_y^2)}{t_y(1-t_x^2)}\cos p \tag{8}$$

and the conditions $\gamma(q), \overline{\gamma}(p) > 0$.

We will call the domain of values of K_x, K_y, where

$$\sinh 2K_x\ \sinh 2K_y > 1$$

(and hence $\gamma(0) = \ln t_x + 2K_y$) the ferromagnetic region of parameters. Notice that in this case the numerator of the integrand in (4) can be represented in the following factorized form in terms of the function $\gamma(q)$:

$$2t_x\left(1+t_y^2\right)+\left(t_y^2-1\right)\left(e^{-ip}+t_x^2e^{ip}\right)=(1-t_y^2)e^{\gamma(0)}(1-e^{-\gamma(\pi)}e^{ip})(1-e^{-\gamma(0)}e^{-ip}).$$

Then, using the identity

$$\frac{1}{N}\sum_q^{(NS)}\frac{1}{\cosh\theta-\cos q}=\frac{\tanh\frac{N\theta}{2}}{\sinh\theta}, \tag{9}$$

one may compute the sum over the discrete Neveu-Schwartz spectrum in (4). As a result, we can express matrix elements $A_{kk'}$ in terms of contour integrals

$$A_{kk'}=\frac{1}{2\pi i}\oint\limits_{|z|=1}\frac{dz}{z}\,z^{k'-k}A(z), \tag{10}$$

$$A(z)=\left[\frac{(1-e^{-\gamma(\pi)}z)(1-e^{-\gamma(0)}z^{-1})}{(1-e^{-\gamma(0)}z)(1-e^{-\gamma(\pi)}z^{-1})}\right]^{\frac{1}{2}}T(z), \tag{11}$$

where

$$T(z)=\tanh\frac{N\overline{\gamma}(p)}{2},\quad z=e^{ip}. \tag{12}$$

Recall that in order to calculate the determinant $|A|$ by the Wiener-Hopf method [3] one has to represent the kernel $A(z)$ in a factorized form

$$A(z)=P(z)Q(z^{-1}), \tag{13}$$

where the functions $P(z)$ and $Q(z)$ are analytic inside the unit circle. Setting

$$W(z)=\frac{P(z)}{Q(z^{-1})} \tag{14}$$

the determinant can be written as follows:

$$|A|=e^{h(r_x)}\sum_{l=0}^{\infty}g_{2l}(r_x), \tag{15}$$

where

$$h(r_x)=r_x\left[\ln P(0)+\ln Q(0)\right]+\frac{1}{2\pi i}\oint\limits_{|z|=1}dz\ln Q(z^{-1})\frac{\partial}{\partial z}\ln P(z), \tag{16}$$

$$g_{2l}(r_x)=\frac{(-1)^l}{l!l!(2\pi i)^l}\oint\limits_{|z_i|<1}\frac{\prod\limits_{i=1}^{2l}(dz_i\,z_i^{r_x})\prod\limits_{i=1}^{l-1}\prod\limits_{j=i+1}^{l}\left[(z_{2i-1}-z_{2j-1})^2(z_{2i}-z_{2j})^2\right]}{\prod\limits_{i=1}^{l}\prod\limits_{j=1}^{l}(1-z_{2i-1}z_{2j})^2}\,\frac{\prod\limits_{i=1}^{l}W(z_{2i-1})}{\prod\limits_{i=1}^{l}W(z_{2i}^{-1})}. \tag{17}$$

We will see that the sum (15) contains a finite number of terms, i. e. there exists l_0 such that $g_{2l} \equiv 0$ for all $l > l_0$.

Let us now calculate the integral (17). To do this, we first rewrite the function $T(z)$ defined by (12) in the factorized form:

$$T(z) = \tanh \frac{N\overline{\gamma}(p)}{2} = \left[\frac{\prod_q^{(R)} (\cosh \overline{\gamma}(p) - \cos q)}{\prod_q^{(NS)} (\cosh \overline{\gamma}(p) - \cos q)} \right]^{\frac{1}{2}} =$$

$$= \left[\frac{\prod_q^{(R)} e^{\gamma(q)} (1 - e^{-\gamma(q)} z)(1 - e^{-\gamma(q)} z^{-1})}{\prod_q^{(NS)} e^{\gamma(q)} (1 - e^{-\gamma(q)} z)(1 - e^{-\gamma(q)} z^{-1})} \right]^{\frac{1}{2}} =$$

$$= P_T(z) Q_T(z^{-1}) = e^{-\Lambda^{-1}} Q_T(z)\, Q_T(z^{-1}).$$

Here, we have introduced the notation

$$\Lambda^{-1} = \frac{1}{2} \left[\sum_q^{(NS)} \gamma(q) - \sum_q^{(R)} \gamma(q) \right],$$

$$Q_T(z) = \left[\frac{\prod_q^{(R)} (1 - e^{-\gamma(q)} z)}{\prod_q^{(NS)} (1 - e^{-\gamma(q)} z)} \right]^{\frac{1}{2}}, \quad P_T(z) = e^{-\Lambda^{-1}}\, Q_T(z). \tag{18}$$

Next let us define

$$P_0(z) = \left[\frac{1 - e^{-\gamma(\pi)} z}{1 - e^{-\gamma(0)} z} \right]^{\frac{1}{2}}, \quad Q_0(z) = \left[\frac{1 - e^{-\gamma(0)} z}{1 - e^{-\gamma(\pi)} z} \right]^{\frac{1}{2}},$$

and write $W(z)$ as follows:

$$\begin{aligned} W(z) &= \frac{P_0(z)}{Q_0(z^{-1})} \frac{P_T(z)}{Q_T(z^{-1})} = \frac{P_0(z)}{Q_0(z^{-1})} \frac{Q_T^2(z)}{T(z)} = \\ &= \frac{t_x(1 - t_y^2)}{t_y(1 - t_x^2)} (\cosh \gamma(\pi) - \cos p)\, Q_T^2(z) \frac{\coth \frac{N\overline{\gamma}(p)}{2}}{\sinh \overline{\gamma}(p)}\, e^{\frac{\gamma(0) - \gamma(\pi)}{2} - 2\Lambda^{-1}}, \end{aligned}$$

where $z = e^{ip}$. Taking into account that

$$\frac{\coth \frac{N\overline{\gamma}(p)}{2}}{\sinh \overline{\gamma}(p)} = \frac{1}{N} \frac{t_y(1 - t_x^2)}{t_x(1 - t_y^2)} \sum_q^{(R)} \frac{1}{\cosh \gamma(q) - \cos p}, \tag{19}$$

one can write $W(z)$ as

$$W(z) = (\cosh \gamma(\pi) - \cos p)\, Q_T^2(z)\, e^{\frac{\gamma(0) - \gamma(\pi)}{2} - 2\Lambda^{-1}}\, \frac{1}{N} \sum_q^{(R)} \frac{1}{\cosh \gamma(q) - \cos p}.$$

Analogous computation for $W^{-1}(z^{-1})$ gives

$$W^{-1}(z^{-1}) = (\cosh\gamma(0) - \cos p)\, Q_T^2(z)\; e^{\frac{\gamma(\pi)-\gamma(0)}{2}}\; \frac{1}{N} \sum_q^{(R)} \frac{1}{\cosh\gamma(q) - \cos p}\,.$$

Using the last two formulas, we can calculate the integral (17). Let us interchange the order of integration and summation over discrete quasimomenta. Then all the singularities inside the integration contours are exhausted by a finite number of poles at the points

$$z_{(k)} = e^{-\gamma(q_{(k)})}, \quad q_{(k)} = \frac{2\pi k}{N}\,, \quad k = 0,\ 1,\ \ldots,\ N-1,$$

corresponding to the Ramond values of quasimomentum. The integral (17) can then be computed by residues, and the result reads

$$g_{2l}(r_x) = \frac{e^{-2l/\Lambda}}{(2l)!N^{2l}} \sum_q^{(R)} \prod_{i=1}^{2l} \frac{e^{-r_x\gamma(q_i)-\eta(q_i)}}{\sinh\gamma(q_i)}\; \mathcal{G}_{2l}[q],$$

where

$$e^{-\eta(q_i)} = \frac{\prod_q^{(R)} \left(1 - e^{-\gamma(q)-\gamma(q_i)}\right)}{\prod_q^{(NS)} \left(1 - e^{-\gamma(q)-\gamma(q_i)}\right)}\,. \tag{20}$$

The function $\mathcal{G}_{2l}[q]$ is given by

$$\begin{aligned}
\mathcal{G}_{2l}[q] \;&=\; C_l^{2l} \prod_{i=1}^{l-1}\prod_{j=i+1}^{l} \left[\left(e^{-\gamma(q_{2i-1})} - e^{-\gamma(q_{2j-1})}\right)^2 \left(e^{-\gamma(q_{2i})} - e^{-\gamma(q_{2j})}\right)^2\right] \times \\
&\times \frac{\prod_{i=1}^{l}\left[\Big(\cosh\gamma(\pi) - \cosh\gamma(q_{2i-1})\Big)\Big(\cosh\gamma(q_{2i}) - \cosh\gamma(0)\Big)\right]}{\prod_{i=1}^{2l} e^{\gamma(q_i)} \prod_{i=1}^{l}\prod_{j=1}^{l}\left(1 - e^{-\gamma(q_{2i-1})-\gamma(q_{2j})}\right)^2} = \\
&= \frac{C_l^{2l}}{2^{2l^2}} \left[\frac{t_y\,(1-t_x^2)}{t_x(1-t_y^2)}\right]^{2l^2} \frac{\prod_{i=1}^{l-1}\prod_{j=i+1}^{l} [(\cos q_{2i-1} - \cos q_{2j-1})^2(\cos q_{2i} - \cos q_{2j})^2]}{\prod_{i<j}^{2l} \sinh^2\frac{\gamma(q_i)+\gamma(q_j)}{2}} \times \\
&\times \prod_{i=1}^{l} [(1+\cos q_{2i-1})(1-\cos q_{2i})]\,.
\end{aligned}$$

It can be symmetrized with respect to the permutations $q_{2i} \leftrightarrow q_{2j-1}$. It turns out that the result of such a symmetrization coincides with the even part of the function

$$\left[\frac{t_y\,(1-t_x^2)}{t_x(1-t_y^2)}\right]^{2l^2} \prod_{i<j}^{2l} \frac{\sin^2\frac{q_i-q_j}{2}}{\sinh^2\frac{\gamma(q_i)+\gamma(q_j)}{2}},$$

so that $g_{2l}(r_x)$ may be written as

$$g_{2l}(r_x) = \frac{e^{-2l/\Lambda}}{(2l)!N^{2l}} \left[\frac{t_y\,(1-t_x^2)}{t_x(1-t_y^2)}\right]^{2l^2} \sum_q^{(R)} \frac{e^{-|r_x|\gamma(q_i)-\eta(q_i)}}{\sinh\gamma(q_i)}\, F_{2l}^2[q], \quad g_0 = 1, \tag{21}$$

$$F_n[q] = \prod_{i<j}^{n} \frac{\sin\frac{q_i-q_j}{2}}{\sinh\frac{\gamma(q_i)+\gamma(q_j)}{2}}\ . \tag{22}$$

The formula (22) implies that for $2l > N$ $F_{2l}[q] \equiv 0$. By virtue of the relation (15), we then obtain the following representation for the correlation function in the ferromagnetic region:

$$\langle\sigma(0,0)\sigma(r_x,0)\rangle^{(-)} = \xi\xi_T\, e^{-|r_x|/\Lambda} \sum_{n=0}^{[N/2]} g_{2n}(r_x), \tag{23}$$

where

$$\xi = \left|1-(\sinh 2K_x \sinh 2K_y)^{-2}\right|^{\frac{1}{4}}, \tag{24}$$

$$\xi_T = \left[\frac{\prod_q^{(R)}\prod_p^{(NS)} \sinh^2\frac{\gamma(q)+\gamma(p)}{2}}{\prod_q^{(R)}\prod_p^{(R)} \sinh\frac{\gamma(q)+\gamma(p)}{2}\ \prod_q^{(NS)}\prod_p^{(NS)} \sinh\frac{\gamma(q)+\gamma(p)}{2}}\right]^{\frac{1}{4}}. \tag{25}$$

3 Paramagnetic phase

In the paramagnetic region of parameters, determined by the condition $\sinh 2K_x\,\sinh 2K_y < 1$, we have the inequality $\ln t_x + 2K_y < 0$, and hence

$$\gamma(0) = -\ln t_x - 2K_y\,.$$

Because of this, the kernel of the Toeplitz matrix whose determinant yields the correlation function should be rearranged:

$$\langle\sigma(0,0)\sigma(r_x,0)\rangle^{(+)} = |A|,$$

$$A_{kk'} = \frac{1}{2\pi i}\oint\limits_{|z|=1} \frac{dz}{z}\, z^{k'-k-1} A(z), \tag{26}$$

$$A(z) = \left[\frac{(1-e^{-\gamma(\pi)}z)(1-e^{-\gamma(0)}z)}{(1-e^{-\gamma(0)}z^{-1})(1-e^{-\gamma(\pi)}z^{-1})}\right]^{\frac{1}{2}} T(z). \tag{27}$$

To calculate the determinant of such a matrix, we must modify the method described in the previous section. For the factorized kernel

$$A(z) = P(z)Q(z^{-1}),$$

where the functions $\ln P(z)$ and $\ln Q(z)$ are analytic inside the unit circle $|z| < 1$, we have the following general formula [1]

$$|A| = \frac{(-1)^{r_x} e^{h(r_x+1)}}{Q^2(0)} \sum_{l=0}^{\infty} f_{2l+1}(r_x), \tag{28}$$

$$f_{2l+1}(r_x) = \frac{(-1)^l}{l!(l+1)!(2\pi i)^{2l+1}} \oint_{|z_i|<1} \prod_{i=1}^{2l+1} (dz_i\, z_i^{r_x}) \frac{\prod_{i=1}^{l} \Big(z_{2i} W(z_{2i}) \Big)}{\prod_{i=0}^{l} \Big(z_{2i+1} W(z_{2i+1}^{-1}) \Big)} \times$$
$$\times \frac{\prod_{i=1}^{l-1} \prod_{j=i+1}^{l} (z_{2i} - z_{2j})^2 \prod_{i=1}^{l} \prod_{j=i+1}^{l+1} (z_{2i-1} - z_{2j-1})^2}{\prod_{i=1}^{l} \prod_{j=0}^{l} (1 - z_{2i} z_{2j+1})^2} . \tag{29}$$

Here, $W(z)$ and $h(r_x)$ are defined by the formulas (14) and (16). Using the transformations analogous to those described above, one finds

$$zW(z) = 2z\, e^{-\frac{\gamma(0)+\gamma(\pi)}{2} - 2/\Lambda} Q_T^2(z)(\cosh\gamma(0) - \cos p)(\cosh\gamma(\pi) - \cos p) \times$$
$$\times \frac{1}{N} \sum_q^{(R)} \frac{1}{\cosh\gamma(q) - \cos p} ,$$
$$z^{-1} W^{-1}(z^{-1}) = \frac{z^{-1}}{2}\, e^{\frac{\gamma(0)+\gamma(\pi)}{2}}\, Q_T^2(z) \frac{1}{N} \sum_q^{(R)} \frac{1}{\cosh\gamma(q) - \cos p} .$$

Using the last two relations, the integral (29) can be computed by residues:

$$f_{2l+1} = \left[\frac{t_y(1-t_x^2)}{t_x(1-t_y^2)} \right]^{2l^2+2l} \frac{e^{-2l/\Lambda + \frac{\gamma(0)+\gamma(\pi)}{2}}}{(2l+1)!N^{2l+1}} \sum_q^{(R)} \prod_{i=1}^{2l+1} \frac{e^{-r_x\gamma(q_i) - \eta(q_i)}}{\sinh\gamma(q_i)} \frac{\mathcal{F}[q]}{\prod_{i<j}^{2l+1} \sinh^2 \frac{\gamma(q_i)+\gamma(q_j)}{2}} ,$$

where

$$\mathcal{F}[q] = \frac{C_l^{2l+1}}{2^{2l(l+1)}} \prod_{i=1}^{l-1} \prod_{j=i+1}^{l} (\cos q_{2i} - \cos q_{2j})^2 \prod_{i=1}^{l} \prod_{j=i+1}^{l+1} (\cos q_{2i-1} - \cos q_{2j-1})^2 \prod_{i=1}^{l} \sin^2 q_{2i} .$$

We again note that after symmetrization over the permutations $q_{2i} \leftrightarrow q_{2j+1}$, the function $\mathcal{F}[q]$ coincides with the even part of the function

$$\prod_{i<j}^{2l+1} \sin^2 \frac{q_i - q_j}{2} .$$

The correlation function in the paramagnetic region of parameters is then given by

$$\langle\sigma(0,0)\sigma(r_x,0)\rangle^{(+)} = \xi\xi_T\, e^{-|r_x|/\Lambda} \sum_{n=0}^{[(N-1)/2]} g_{2n+1}(r_x), \tag{30}$$

$$g_{2l+1}(r_x) = \frac{e^{-(2l+1)/\Lambda}}{(2l+1)!N^{2l+1}} \left[\frac{t_y\,(1-t_x^2)}{t_x(1-t_y^2)}\right]^{(2l+1)^2/2} \sum_q{}^{(R)} \frac{e^{-|r_x|\gamma(q_i)-\eta(q_i)}}{\sinh\gamma(q_i)}\, F_{2l+1}^2[q], \tag{31}$$

$$F_n[q] = \prod_{i<j}^{n} \frac{\sin\frac{q_i-q_j}{2}}{\sinh\frac{\gamma(q_i)+\gamma(q_j)}{2}}\,, \quad F_1 = 1. \tag{32}$$

Comparing the formulas (21)–(25) and (30)–(32) for the correlation function with the isotropic case, we see that all the difference consists in the redefinition of the function $\gamma(q)$ given by (7) and in the appearance of the multipliers

$$\frac{t_y(1-t_x^2)}{t_x(1-t_y^2)}$$

in the appropriate powers in the function $g_n(r_x)$. In the thermodynamic limit $N\to\infty$, these expressions reduce to the known formulas [4].

4 Magnetic susceptibility

Since we have two independent coupling parameters in the anisotropic Ising model on the cylinder, we can consider two limit cases:

$$K_x \to 0, \qquad K_y \neq 0$$

and

$$K_x \neq 0, \qquad K_y \to 0.$$

The two-dimensional Ising lattice splits into mutually non-interacting closed one-row Ising chains of the length N in the first case, and into N chains of infinite length in the second case. Such limit cases provide primitive tests for the form factor representations obtained above. To perform such checks, it would also be desirable to have an expression for the correlation function $\langle\sigma(0)\sigma(\mathbf{r})\rangle$ of two spins with arbitrary location on the lattice, i. e. for $r_x \neq 0$ and $r_y \neq 0$. The same problem arises when one tries to write the lattice analog of the Lehmann representation for the two-point Green function or to compute the magnetic susceptibility: in both cases we have to perform the Fourier transformation $\sum_{\mathbf{q}} e^{i\mathbf{qr}}\langle\sigma(0)\sigma(\mathbf{r})\rangle$. Meanwhile, the matrix whose determinant gives the correlation function has a non-Toeplitz form for $r_y \neq 0$, and the method used above to find form factor expansions becomes inapplicable. The answer can nevertheless be obtained in this case as well if we formulate the problem in terms of spin matrix elements.

Recall that the 2^N-dimensional space in which the Ising transfer matrix $\mathcal{T}$ acts can be splitted into two invariant subspaces — the Neveu-Schwartz sector (NS) and the Ramond

sector (R). The eigenvalues corresponding to the eigenvectors from each subspace are given by

$$\lambda^{(NS)} = (2\sinh K_x)^{N/2} \exp \frac{1}{2}\left\{ \pm\gamma\left(\frac{\pi}{N}\right) \pm \gamma\left(\frac{3\pi}{N}\right) \pm \ldots \pm \gamma\left(2\pi - \frac{\pi}{N}\right)\right\}, \quad (33)$$

$$\lambda^{(R)} = (2\sinh K_x)^{N/2} \exp \frac{1}{2}\left\{ \pm\gamma(0) \pm \gamma\left(\frac{2\pi}{N}\right) \pm \ldots \pm \gamma\left(2\pi - \frac{2\pi}{N}\right)\right\}, \quad (34)$$

and we have different selection rules in different regions of parameters of the model: the number of minus signs in the NS–sector (33) is even in both ferromagnetic and paramagnetic region, while the number of minus signs in the R–sector (34) is even in the ferromagnetic region and odd in the paramagnetic one. Note that the expressions (33), (34) for the tranfer matrix eigenvalues can be obtained from the representation (1) for the partition function, if we expand the products (5):

$$Z = Z^{(NS)} + Z^{(R)},$$

$$Z^{(NS)} = \frac{1}{2}(Q^{(NS,NS)} + Q^{(R,NS)}) = \sum_i (\lambda_i^{NS})^M,$$

$$Z^{(R)} = \frac{1}{2}(Q^{(NS,R)} - Q^{(R,R)}) = \sum_i (\lambda_i^{R})^M.$$

Because of translation invariance, the transfer matrix can be diagonalized simultaneously with the translation operator. This additional requirement determines the orthonormal basis of eigenvectors up to permutations. It is convenient to interpret the elements of this basis in terms of NS– and R–multiparticle states with appropriate values of quasimomenta: they are generated from the vacua $|\emptyset\rangle_{NS}$ and $|\emptyset\rangle_R$ by the action of fermionic creation operators.

Let us consider for definiteness the ferromagnetic region and show how to express the two-point correlation function in terms of the matrix elements

$${}_{NS}\langle q_1, \ldots, q_{2K}|\hat\sigma(0,0)|p_1, \ldots, p_{2L}\rangle_R$$

of Ising spin (the elements NS–NS and R–R vanish due to $\mathbb{Z}_2$-symmetry of the model):

$$\begin{aligned}
\langle \sigma(0,0)\sigma(r_x,r_y)\rangle &= {}_{NS}\langle\emptyset|\hat\sigma(0,0)\hat\sigma(r_x,r_y)|\emptyset\rangle_{NS} = \\
&= {}_{NS}\langle\emptyset|\hat\sigma(0,0)\mathcal{T}^{r_x}\hat\sigma(0,r_y)\mathcal{T}^{-r_x}|\emptyset\rangle_{NS} = \\
&= \sum_{K=0}^{[N/2+1]} \frac{e^{-r_x/\Lambda}}{(2K)!} \sum_{p_1\ldots p_{2K}}^{(R)} |{}_{NS}\langle\emptyset|\hat\sigma(0,0)|p_1\ldots p_{2K}\rangle_R|^{\,2} \times \\
&\qquad \times\ \exp\left(-r_x\sum_{j=1}^{2K}\gamma(p_j) + ir_y\sum_{j=1}^{2K}p_j\right). \qquad (35)
\end{aligned}$$

Comparing this expression with the formulas (21)–(23), we find spin matrix elements between the NS-vacuum and an arbitrary R–eigenstate:

$$\left|{}_{NS}\langle\emptyset|\hat{\sigma}(0,0)|p_1\dots p_L\rangle_R\right|^2 = \xi\xi_T \left[\frac{t_y\,(1-t_x^2)}{t_x(1-t_y^2)}\right]^{L^2/2} \prod_{j=1}^{L} \frac{e^{-\eta(p_j)-\Lambda^{-1}}}{N\sinh\gamma(p_j)} \prod_{i<j}^{L} \frac{\sin^2\frac{p_i-p_j}{2}}{\sinh^2\frac{\gamma(p_i)+\gamma(p_j)}{2}}\,. \tag{36}$$

The formula (35) then implies that form factor expansion of the correlation function $\langle\sigma(0,0)\sigma(r_x,r_y)\rangle$ is obtained from (21)–(23) by the substitution

$$\frac{e^{-|r_x|\gamma(q_i)-\eta(q_i)}}{\sinh\gamma(q_i)} \to \frac{e^{-|r_x|\gamma(q_i)+ir_yq_i-\eta(q_i)}}{\sinh\gamma(q_i)}$$

in the function g_n. An analogous result may be found in the paramagnetic case as well. Thus we have

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle^{(-)} = \xi\xi_T\, e^{-|r_x|/\Lambda} \sum_{l=0}^{[N/2]} g_{2l}(r_x,r_y), \tag{37}$$

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle^{(+)} = \xi\xi_T\, e^{-|r_x|/\Lambda} \sum_{l=0}^{[(N-1)/2]} g_{2l+1}(r_x,r_y), \tag{38}$$

where

$$g_n(r_x,r_y) = \frac{e^{-n/\Lambda}}{n!N^n} \left[\frac{t_y(1-t_x^2)}{t_x(1-t_y^2)}\right]^{n^2/2} \sum_q^{(R)} \prod_{j=1}^{n} \frac{e^{-|r_x|\gamma(q_j)+ir_yq_j-\eta(q_j)}}{\sinh\gamma(q_j)}\; F_n^2[q]\,. \tag{39}$$

As an illustration, consider the “paramagnetic” expansion for $K_y=0$. Notice that in this limit

$$\cosh\gamma(q) = \coth 2K_x, \quad \gamma(q) = t_x = \text{const},$$

and, therefore

$$\xi = \xi_T = 1, \quad \Lambda^{-1} = \eta(q) = 0,$$

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle = \frac{1}{N}\sum_q^{(b)} (t_x)^{|r_x|} e^{ir_yq} = (t_x)^{|r_x|}\,\delta_{0\,r_y}.$$

As expected, the spins from different lattice rows are uncorrelated.

Magnetic susceptibility of the two-dimensional Ising model on a finite $M\times N$ lattice in zero field may be written as a sum of correlation functions:

$$\beta^{-1}\chi = \sum_{r_x=0}^{M-1}\sum_{r_y=0}^{N-1}\langle\sigma(0,0)\sigma(r_x,r_y)\rangle.$$

In order to find the susceptibility on the cylinder, we can use the expression

$$\beta^{-1}\chi = \sum_{r_x=-\infty}^{\infty}\sum_{r_y=0}^{N-1}\langle\sigma(0,0)\sigma(r_x,r_y)\rangle$$

with some precautions [2]. Because of simple structure of the form factor expansions (37)–(39) this sum is easily computed, and we obtain the following expressions for the susceptibility in the ferromagnetic and paramagnetic region of parameters:

$$\beta^{-1}\chi^{(-)} = \xi\xi_T \sum_{l=0}^{[N/2]} \chi_{2l}, \qquad \beta^{-1}\chi^{(+)} = \xi\xi_T \sum_{l=0}^{[(N-1)/2]} \chi_{2l+1}\,,$$

where

$$\chi_n = \frac{e^{-n/\Lambda}}{n!N^{n-1}} \left[\frac{t_y(1-t_x^2)}{t_x(1-t_y^2)}\right]^{n^2/2} \times$$

$$\times \sum_q{}^{(R)} \prod_{j=1}^{n} \frac{e^{-\eta(q_j)}}{\sinh\gamma(q_j)}\, F_n^2[q]\, \coth\left(\frac{1}{2}\sum_{j=1}^{n}\gamma(q_j) + \Lambda^{-1}\right)\delta\left(\sum_{j=1}^{n} q_j,\ 0 \mod 2\pi\right),$$

and δ is the Kronecker symbol.

5 Scaling limit

An important stage in the study of the two-dimensional Ising model is the analysis of its scaling limit [5, 6]. A new effect with respect to the case of the infinite plane is that the "cylindrical parameters" ξ_T, Λ^{-1}, η, which tend to zero for fixed distance from the critical point ($N \to \infty$, $\sinh 2K - 1 = \text{const} \neq 0$), do not vanish in the scaling limit on the cylinder. Although the formulas for correlation functions obviously become more involved, it is possible to generalize several results obtained for the infinite plane to the case of the isotropic Ising model on the cylinder [7]. For instance, it was shown [8] that the Ising model correlation functions on the cylinder satisfy some integrable equations generalizing Painlevé III and Painlevé V equations obtained in the planar case. When considering the scaling limit of the anisotropic model on the cylinder, we do not encounter much difficulty provided we take into account some subtleties in the definition of scaling variables.

Since in this case there are two parameters K_x, K_y, instead of a critical point we have the critical line $\sinh 2K_x \sinh 2K_y - 1 = 0$. One has certain freedom in the definition of the distance from the critical line, which can be used to ensure that the results obtained in the anisotropic case in the scaling limit coincide with those obtained for the isotropic ($K_x = K_y$) model. It turns out that this condition is satisfied with the following choice of scaling variables:

$$|r_x| \to \infty, \quad |r_y| \to \infty, \quad N \to \infty, \quad \gamma(0) \to 0,$$

$$\gamma(0)r_x = x = \text{const}, \quad \gamma(0)r_y \sinh 2K_x = y = \text{const}, \quad \gamma(0)N \sinh 2K_x = \beta = \text{const}. \quad (40)$$

Because of the presence of the exponential factors $\mathrm{e}^{-|r_x|\gamma(q)+ir_yq}$ and $\mathrm{e}^{-N\gamma(q)}$ in the corresponding sums and integrals, we can suppose that $q \ll 1$ and, therefore,

$$\gamma(q) = \gamma(0)\sqrt{1+\left(\frac{q}{\gamma(0)\sinh 2K_x}\right)^2}\,, \quad \overline{\gamma}(q) = \gamma(0)\sinh 2K_x\sqrt{1+\left(\frac{q}{\gamma(0)}\right)^2}\,.$$

In order to find the scaling limit asymptotics of the cylindrical parameters Λ, ξ_T and $\eta(q)$, it is convenient to use integral representations

$$\begin{aligned}
\Lambda^{-1} &= \frac{1}{\pi}\int_0^\pi dp\ \ln\coth\frac{N\overline{\gamma}(p)}{2}\,, \\
\eta(q) &= \frac{1}{\pi}\int_0^\pi dp\ \frac{\cos p - e^{-\gamma(q)}}{\cosh\gamma(q) - \cos p}\ \ln\coth\frac{N\overline{\gamma}(p)}{2}\,, \\
\xi_T &= \frac{N^2}{2\pi^2}\int_0^\pi \frac{dp\, dq\ \overline{\gamma}\,'(p)\ \overline{\gamma}\,'(q)}{\sinh N\overline{\gamma}(p)\ \sinh N\overline{\gamma}(q)}\ \ln\left|\frac{\sin(p+q)/2}{\sin(p-q)/2}\right|,
\end{aligned}$$

equivalent to the formulas (18), (20), (25). Denoting

$$\omega(q) = \sqrt{1+q^2}$$

and using the above relations, we obtain the answer for the scaled correlation function of the 2D anisotropic Ising model on the cylinder:

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle^{(-)} = \xi\tilde{\xi}_T(\beta)\, e^{-|x|/\tilde{\Lambda}(\beta)}\ \sum_{n=0}^{\infty}\tilde{g}_{2n}(x,y,\beta)\,, \tag{41}$$

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle^{(+)} = \xi\tilde{\xi}_T(\beta)\, e^{-|x|/\tilde{\Lambda}(\beta)}\ \sum_{n=0}^{\infty}\tilde{g}_{2n+1}(x,y,\beta)\,, \tag{42}$$

$$\tilde{g}_n(x,y,\beta) = \frac{1}{n!\beta^n}\sum_{[l]}\prod_{j=1}^{n}\frac{e^{-|x|\omega(q_j)+iyq_j-\tilde{\eta}(q_j,\beta)}}{\omega(q_j)}\ \ \tilde{F}_n^2[l]\,. \tag{43}$$

Here, the summation is performed over integer l

$$\sum_{[l]} = \sum_{l_1=-\infty}^{\infty}\cdots\sum_{l_n=-\infty}^{\infty}\,,$$

the quasimomenta take on the bosonic values $q_j = \frac{2\pi l_j}{\beta}$, and the cylindrical parameters and form factors are given by

$$\begin{aligned}
\tilde{\Lambda}^{-1}(\beta) &= \frac{1}{\pi}\int_0^\infty dp\ \ln\coth\frac{\beta\omega(p)}{2}\,, \\
\tilde{\eta}(q_j,\beta) &= \frac{1}{\pi}\int_0^\infty dp\ \frac{2\omega(q_j)}{\omega^2(q_j)+p^2}\ \ln\coth\frac{\beta\omega(p)}{2}\,, \\
\tilde{\xi}_T(\beta) &= \frac{\beta^2}{2\pi^2}\int_0^\infty \frac{dp\, dq\ \omega\,'(p)\ \omega\,'(q)}{\sinh\beta\,\omega(p)\ \sinh\beta\omega(q)}\ \ln\left|\frac{p+q}{p-q}\right|,
\end{aligned}$$

$$\tilde{F}_n[l] = \prod_{1\le i<j\le n} \frac{q_i - q_j}{\omega(q_i) + \omega(q_j)} \,.$$

We see that the spin-spin correlation function on the cylinder in the scaling limit defined by (40) is given by the same functions of renormalized coordinates as in the isotropic model. Therefore, all results of [8] (determinant representations of the correlation fuctions, differential equations) apply to the anisotropic case as well.

6 Spin matrix elements

In order to calculate multipoint correlation functions on the cylinder and torus, it is necessary (and sufficient) to have formulas not only for the form factors $|{}_{NS}\langle\emptyset|\hat{\sigma}(0,0)|p_1\dots p_L\rangle_R|$, but also for all other spin matrix elements. For instance, the two-point correlation function in the Ising model on the periodic lattice of size $M\times N$ in the ferromagnetic region of parameters can be written as follows:

$$\langle\sigma(0,0)\sigma(r_x,r_y)\rangle = \frac{\mathrm{Tr}\left\{\hat{\sigma}(0,0)\mathcal{T}^{r_x}\hat{\sigma}(0,r_y)\mathcal{T}^{M-r_x}\right\}}{\mathrm{Tr}\,\mathcal{T}^M} =$$

$$= \sum_{K=0}^{[N/2+1]}\sum_{L=0}^{[N/2+1]}\sum_{q_1\dots q_{2K}}^{(NS)}\sum_{p_1\dots p_{2L}}^{(R)} \left|{}_{NS}\langle q_1\dots q_{2K}|\hat{\sigma}(0,0)|p_1\dots p_{2L}\rangle_R\right|^2 \frac{e^{ir_y\sum\limits_{j=1}^{2K} q_j - ir_y\sum\limits_{j=1}^{2L} p_j}}{(2K)!(2L)!}\times$$

$$\times\left\{e^{-(M-r_x)\sum\limits_{j=1}^{2K}\gamma(q_j)-r_x\left(\Lambda^{-1}+\sum\limits_{j=1}^{2L}\gamma(p_j)\right)} + e^{-r_x\sum\limits_{j=1}^{2K}\gamma(q_j)-(M-r_x)\left(\Lambda^{-1}+\sum\limits_{j=1}^{2L}\gamma(p_j)\right)}\right\}\Big/\ \mathrm{Tr}\left(\frac{\mathcal{T}}{\lambda_0}\right)^M,$$

where λ_0 denotes the largest transfer matrix eigenvalue, which corresponds to the eigenvector $|\emptyset\rangle_{NS}$.

In [9], we have proposed a formula for arbitrary spin matrix elements of the isotropic Ising model on a finite periodic lattice. This formula can be easily generalized to the anisotropic case, using two hints from the above:

- assume that all the difference due to anisotropy consists in the redefinition of the function $\gamma(q)$ in the formula for spin matrix element, and in the appearance of the factor $\dfrac{t_y(1-t_x^2)}{t_x(1-t_y^2)}$ to some power (see, e. g., the formula (36));
- also assume that in the scaling limit (40) *all* multipoint correlation functions are given by the same formulas [7, 9] as in the isotropic case.

Omitting the calculations, we present only the final expression for the spin matrix element, which follows from the above assumptions:

$$\left|{}_{NS}\langle q_1\dots q_K|\hat{\sigma}(0,0)|p_1\dots p_L\rangle_R\right|^2 = \xi\xi_T\prod_{j=1}^{K}\frac{e^{\eta(q_j)+\Lambda^{-1}}}{N\sinh\gamma(q_j)}\prod_{j=1}^{L}\frac{e^{-\eta(p_j)-\Lambda^{-1}}}{N\sinh\gamma(p_j)}\times$$

$$\times \left[\frac{t_y\,(1-t_x^2)}{t_x(1-t_y^2)}\right]^{\frac{(K-L)^2}{2}} \prod_{i<j}^{K} \frac{\sin^2\frac{q_i-q_j}{2}}{\sinh^2\frac{\gamma(q_i)+\gamma(q_j)}{2}} \prod_{i<j}^{L} \frac{\sin^2\frac{p_i-p_j}{2}}{\sinh^2\frac{\gamma(p_i)+\gamma(p_j)}{2}} \prod_{\substack{1\le i\le K\\ 1\le j\le L}} \frac{\sinh^2\frac{\gamma(q_i)+\gamma(p_j)}{2}}{\sin^2\frac{q_i-p_j}{2}}\,. \quad (44)$$

As in the isotropic model case, this representation was verified for finite-row Ising chains with $N = 1, 2, 3, 4$. The scaling limit of the expression (44) obviously coincides with the classical result [11] in the limit of continuous infinite plane, and with the corresponding results on the continuous infinite cylinder [10].

7 Discussion

Lattice systems with cylindrical geometry recently became important for nanoelectronics. However, the elementary cell in these systems is hexagonal. The star-triangle transformation brings the Ising model defined on such a lattice to the Ising model on a triangular lattice. The simple form of the expressions obtained in the present paper gives a hope that similar results can also be obtained in the latter case.

Exact expressions for the correlation functions are also known for several 1D quantum systems, in particular, for the XXZ Heisenberg spin chain [12]. In these models, matrix elements of local operators on the hamiltonian eigenstates were calculated using the Bethe ansatz. One can try to use our results to analyze the behaviour of such integrable one-dimensional models in a finite volume and/or at non-zero temperature.

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